Limit Techniques Practice - 10/14/16

1. Evaluate $\lim_{x \to 0} x^3 \cos\left(\frac{1}{x}\right)$.

   Solution: Note that the range of cos is $[-1, 1]$, so $-1 \le \cos(1/x) \le 1$. Multiplying through by x^3 gives us $-x^3 \le x^3 \cos(1/x) \le x^3$ if x is positive. Notice that $\lim_{x \to 0^+} -x^3 = \lim_{x \to 0^+} x^3 = 0$, so by the Squeeze Theorem, $\lim_{x \to 0^+} x^3 \cos(1/x) = 0$. If x is negative, the inequalities switch when we multiply through by x^3, so we have $x^3 \le x^3 \cos(1/x) \le -x^3$. But again, $\lim_{x \to 0^-} x^3 = 0 = \lim_{x \to 0^-} -x^3$, so by the Squeeze Theorem, $\lim_{x \to 0^-} x^3 \cos(1/x) = 0$. Since the left and right hand limits are the same, then we have that the overall limit is 0.

2. Evaluate $\lim_{x \to 7} \frac{\sqrt{x+2} - 3}{x - 7}$.

   Solution: We want to multiply top and bottom by $\sqrt{x+2}+3$ to get $\lim_{x \to 7} \frac{(\sqrt{x+2}-3)(\sqrt{x+2}+3)}{(x-7)(\sqrt{x+2}+3)} = \lim_{x \to 7} \frac{x+2-9}{(x-7)(\sqrt{x+2}+3)} = \lim_{x \to 7} \frac{1}{\sqrt{x+2}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}$.

3. Evaluate $\lim_{x \to 0} x \arctan\left(\frac{1}{x^2}\right)$.

   Solution: Since the range of arctan is $[-\frac{\pi}{2}, \frac{\pi}{2}]$, so $-\frac{\pi}{2} \le \arctan(1/x^2) \le \frac{\pi}{2}$. Multiplying through by x gives $-\frac{\pi x}{2} \le x \arctan(1/x^2) \le \frac{\pi x}{2}$ if x is positive. Then $\lim_{x \to 0^+} -\frac{\pi x}{2} = \lim_{x \to 0^+} \frac{\pi x}{2} = 0$, so by the Squeeze Theorem, $\lim_{x \to 0^+} x \arctan\left(\frac{1}{x^2}\right) = 0$. If x is negative, then the inequalities flip when we multiply through by x, so we get $\frac{\pi x}{2} \le x \arctan(1/x^2) \le -\frac{\pi x}{2}$. Again $\lim_{x \to 0^-} \frac{\pi x}{2} = \lim_{x \to 0^-} -\frac{\pi x}{2} = 0$, so by the Squeeze Theorem, we get that $\lim_{x \to 0^-} x \arctan(1/x^2) = 0$. Since the right and left hand limits are the same, we have that the overall limit is 0. Notice that if we were multiplying through by an even power of x, we would not have to break our calculations into two cases, since an even power of x will always be a positive number regardless of what x is.

4. Evaluate $\lim_{x \to 0} \left(\frac{3}{x\sqrt{9-x}} - \frac{1}{x}\right)$.

   Solution: We first want to combine the fractions in the limit to get $\frac{3}{x\sqrt{9-x}} - \frac{\sqrt{9-x}}{x\sqrt{9-x}} = \frac{3-\sqrt{9-x}}{x\sqrt{9-x}}$. We want to eventually get rid of the x in the denominator, since that is what is preventing us from just plugging in 0 to find the limit. Thus we will multiply the top and bottom by the conjugate of the numerator to get $\frac{9-(9-x)}{x\sqrt{9-x}(3+\sqrt{9-x})} = \frac{-x}{x\sqrt{9-x}(3+\sqrt{9-x})}$. Now when we take the limit, we can cancel the x on top and bottom to get $\lim_{x \to 0} \frac{-1}{\sqrt{9-x}(3+\sqrt{9-x})} = \frac{-1}{3(3+3)} = \frac{-1}{18}$.

5. Evaluate $\lim\limits_{x \to 2} \dfrac{\sqrt{6-x}-2}{\sqrt{3-x}-1}$.

Solution: We want to multiply by a conjugate, the question is just, by which one? It turns out that it doesn't matter. Let's start by multiplying top and bottom by $\sqrt{6-x}+2$ (the conjugate of the numerator) to get $\frac{6-x-4}{(\sqrt{3-x}-1)(\sqrt{6-x}+2} = \frac{2-x}{(\sqrt{3-x}-1)(\sqrt{6-x}+2)}$. Now we will multiply top and bottom by the conjugate of the denominator, $\sqrt{3-x}+1$. This gives us $\frac{(2-x)(\sqrt{3-x}+1)}{(3-x-1)(\sqrt{6-x}+2)} = \frac{(2-x)(\sqrt{3-x}+1)}{(2-x)(\sqrt{6-x}+2)}$. Now we can cancel the $2-x$ when we take the limit, so we have $\lim_{x \to 2} \frac{\sqrt{3-x}+1}{\sqrt{6-x}+2} = \frac{1+1}{2+2} = \frac{1}{2}$.